ON SOME SPECIAL SOLUTIONS OF THE PROBLEM OF MOTION OF A HEAVY RIGID BODY ABOUT A FIXED POINT

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1. Reduction of the problem to the solution of two differential equations when the center of gravity of the body is located on one of the principal axes of inertia. The problem of motion of a heavy rigid body about a fixed point, with the center of gravity of the body located on one of the principal axes of inertia (e.g., $y_0 = z_0 = 0$), is reducible [1] to the differential equations

$$\left(\frac{d\nu}{dt}\right)^2 = -\nu(\tau - h)^2 + 4Q\,[Q(\nu - \rho^2) + k\rho(\tau - h) - Qk^2] \tag{1.1}$$
$$A^2BC\left(\frac{d\rho}{dt}\right)^2 = -A^2\nu^2 + AJ\nu\rho^2 + A^2B_1\nu\tau + P'\rho^4 + AN\rho^2\tau - A^2BC\tau^2$$
$$(\rho^2 - A\tau)\frac{d\nu}{dt} + A\,[\rho(\tau - h) - 2Qk]\frac{d\rho}{dt} + A(\nu - \rho^2)\frac{d\tau}{dt} = 0$$

Here

$$\nu = A^2p^2 + B^2q^2 + C^2r^2, \qquad \rho = \frac{1}{R_0}(Ax_0p + By_0q + Cz_0r) = Ap$$
$$\tau = Ap^2 + Bq^2 + Cr^2 \tag{1.2}$$
$$B_1 = B + C, \qquad J = 2A - B - C, \qquad N = 2BC - AB - AC$$
$$P' = (A - B)(C - A), \quad Q = Mgx_0$$

and k, h, respectively, are constants of the integral of the moment of momentum relative to the vertical axis and the integral of the kinetic energy, which are defined by formulas (1.3) and (2.6) of [1]; the rest of the notation is the usual one.

The variables γ_1, γ_2, γ_3 are defined by equations (2.9) of [1] :

$$H_0\gamma_1 = \sqrt{H}W_{\gamma_1} - H_kAp - H_\mu$$
$$H_0\gamma_2 = \sqrt{H}W_{\gamma_2} - H_kBq \tag{1.3}$$
$$H_0\gamma_3 = \sqrt{H}W_{\gamma_3} - H_kCr$$

where according to (2.6), (2.7), and (2.8) of [1],

$$H_0 = \nu - \rho^2 \neq 0, \qquad \sqrt{H} = \frac{1}{2Q}\frac{d\nu}{dt}$$

$$H_k = \rho\mu - k = \frac{1}{2Q}[\rho(\tau - h) - 2Qk], \quad H_\mu = k\rho - \nu\mu = -\frac{1}{2Q}[\nu(\tau - h) - 2Qk\rho]$$

$$W_{\gamma_1} = 0, \qquad W_{\gamma_2} = Cr, \qquad W_{\gamma_3} = -Bq$$

Equations (1.3) assume the form

$$2Q\gamma_1 = \tau - h$$
$$2Q(\nu - \rho^2)\gamma_2 = Cr\frac{d\nu}{dt} - Bq[\rho(\tau - h) - 2Qk] \tag{1.4}$$
$$2Q(\nu - \rho^2)\gamma_3 = -Bq\frac{d\nu}{dt} - Cr[\rho(\tau - h) - 2Qk]$$

Eliminating dt from equations (1.1), we obtain

$$\left(\frac{d\nu}{d\rho}\right)^2 = f(\nu, \rho, \tau, k, h) \tag{1.5}$$
$$(\rho^2 - A\tau)\frac{d\nu}{d\rho} + A(\nu - \rho^2)\frac{d\tau}{d\rho} + A[\rho(\tau - h) - 2Qk] = 0$$

Substituting the first Euler equation

$$\frac{d\nu}{dt} = (B - C)qr\frac{d\nu}{d\rho}$$

into (1.4) and taking into account the relations

$$\nu - \rho^2 = -\frac{B}{A}(\rho^2 - A\tau) - C(B - C)r^2$$
$$\nu - \rho^2 = -\frac{C}{A}(\rho^2 - A\tau) + B(B - C)q^2$$

and the second equation of (1.5), we get

$$2Q\gamma_1 = \tau - h$$
$$2Q\gamma_2 = q\left(-\frac{d\nu}{d\rho} + B\frac{d\tau}{d\rho}\right) \tag{1.6}$$
$$2Q\gamma_3 = r\left(-\frac{d\nu}{d\rho} + C\frac{d\tau}{d\rho}\right)$$

The system of differential equations (1.5), where f is a known function, can be integrated by writing ν and r as polynomials in ρ and equating the coefficients of equal powers of ρ. The quantities p, q, r, according to (1.2), can also be expressed explicitly as functions of ρ (under the conditions obtained here).

The problem is reduced to one quadrature, that of the second differential equation of (1.1), with ν, r replaced by the appropriate functions of ρ.

This quadrature introduces a single arbitrary constant t_0, which we shall simply add to t.

We assume that one of the necessary conditions, $k = 0$, for the integrability of the Euler-Poisson equations, obtained in [1] is satisfied.

From this condition, or from the second equation of (1.5), we then obtain

$$2Q\gamma_1 = -\frac{1}{\rho}\left[\left(\frac{\rho^2}{A} - \tau\right)\frac{d\nu}{d\rho} + (\nu - \rho^2)\frac{d\tau}{d\rho}\right] \tag{1.7}$$

Hence γ_1, γ_2, γ_3 are expressed as explicit functions of ρ by means of (1.6), (1.7).

2. Steklov's second case. To simplify the calculations, we shall use some of the results of [1].

We suppose that ν and τ can be represented in the form of polynomials of degree n in ρ with constant coefficients

$$\begin{aligned} \nu &= a_0 + a_1\rho + a_2\rho^2 + \ldots + a_n\rho^n \\ \tau &= b_0 + b_1\rho + b_2\rho^2 + \ldots + b_n\rho^n \end{aligned} \tag{2.1}$$

Expressions for the initial coefficients of the expansions of the functions $\nu(t)$, $\tau(t)$ in a neighborhood of a pole of the second order and of $\rho(t)$ in a neighborhood of a pole of order one

$$\begin{aligned} \nu &= t^{-2}(\nu_0 + \nu_1 t + \nu_2 t^2 + \ldots) \\ \rho &= t^{-1}(\rho_0 + \rho_1 t + \rho_2 t^2 + \ldots) \\ \tau &= t^{-2}(\tau_0 + \tau_1 t + \tau_2 t^2 + \ldots) \end{aligned} \tag{2.2}$$

were given in Section 4 of [1].

One of the necessary conditions for the special cases of the integrability of the Euler-Poisson equations, $k = 0$, is obtained from these expansions. We suppose that this condition is satisfied and that the coefficients of the expansions have the values

$$\begin{gathered} \nu_0 = -A^2, \quad \rho_0 = iA\sqrt{\frac{(A-2B)(A-2C)}{(A-B)(A-C)}} \qquad \tau_0 = -2A \\ \nu_1 = \rho_1 = \tau_1 = 0 \end{gathered} \tag{2.3}$$

Substituting (2.2) into (2.1), equating the coefficients of like powers of t, and using (2.3), we obtain

$$a_2 = \frac{(A-B)(A-C)}{(A-2B)(A-2C)}, \quad a_0 = \nu_2 - 2iA\sqrt{\frac{(A-B)(A-C)}{(A-2B)(A-2C)}}\rho_2,$$

$$b_2 = \frac{2(A-B)(A-C)}{A(A-2B)(A-2C)}, \quad b_0 = \tau_2 - 4i\sqrt{\frac{(A-B)(A-C)}{(A-2B)(A-2C)}}\rho_2$$

$$a_1 = b_1 = 0, \quad a_i = b_i = 0 \quad (i = 3, 4, 5, \ldots)$$

Hence the formulas (2.1) take the form

$$\nu = a_0 + \frac{(A-B)(A-C)}{(A-2B)(A-2C)}\rho^2, \quad \tau = b_0 + \frac{2(A-B)(A-C)}{A(A-2B)(A-2C)}\rho^2 \tag{2.4}$$

Substituting (2.4) in (1.6) and (1.7), we get

$$\gamma_1 = \frac{(A-B)(A-C)}{QA(A-2B)(A-2C)}[B(A-2B)q^2 + C(A-2C)r^2]$$
$$\gamma_2 = \frac{(A-B)(A-C)}{Q(2C-A)}pq, \quad \gamma_3 = \frac{(A-B)(A-C)}{Q(2B-A)}pr \tag{2.5}$$

If the polynomials (2.4) for ν and τ are substituted in the first and third expressions of (1.2), with $\rho = Ap$, and the resulting two equations are solved for p^2 and q^2, we obtain the expressions for p^2 and q^2 in terms of r^2 which were found by Steklov.

We have arrived at Steklov's second case [2]. In this connection, we may mention that not only Steklov, but also Field [3], Gorliss [4], Fabbri [5,6,7] and Kuz'min [8,9] have been concerned with integrating the equations and clarifying the various possiblities of the motion.

3. Goriachev's second case. We perform the transformation

$$t = \xi^2 \tag{3.1}$$

on the independent variable in equations (1.1).

As a result, the system (1.1) becomes

$$\left(\frac{d\nu}{d\xi}\right) = 4\xi^2\{-\nu(\tau-h)^2 + 4Q[Q(\nu-\rho^2) + k\rho(\tau-h) - Qk^2]\}$$
$$A^2BC\left(\frac{d\rho}{d\xi}\right)^2 = 4\xi^2[-A^2\nu^2 + AJ\nu\rho^2 + A^2B_1\nu\tau + P'\rho^4 + AN\rho^2\tau - A^2BC\tau^2]$$
$$(\rho^2 - A\tau)\frac{d\nu}{d\xi} + A[\rho(\tau-h) - 2Qk]\frac{d\rho}{d\xi} + A(\nu-\rho^2)\frac{d\tau}{d\xi} = 0 \tag{3.2}$$

This system of equations admits expansions of the functions $\nu(\xi)$, $\rho(\xi)$, $\tau(\xi)$ about a pole different from (2.2):

$$\nu = \xi^{-4}(u_0 + u_1\xi + u_2\xi^2 + \ldots) \qquad (u_0 \neq 0)$$
$$\rho = \xi^{-1}(v_0 + v_1\xi + v_2\xi^2 + \ldots) \qquad (v_0 \neq 0) \tag{3.3}$$
$$\tau = \xi^{-4}(w_0 + w_1\xi + w_2\xi^2 + \ldots) \qquad (w_0 \neq 0)$$

Substituting (3.3) into (3.2), equating coefficients of equal powers of ξ, and assuming the condition $k = 0$ to be satisfied, for determining the coefficients of the series (3.3) we obtain the following systems of equations:

$$1. \qquad 4u_0 + w_0^2 = 0, \qquad u_0(u_0 - B_1 w_0) + BCw_0^2 = 0 \tag{3.4}$$

(The third equation is an identity).

$$2. \qquad u_1 + w_1 w_0 = 0, \qquad u_1 Y_0 - w_1 \Gamma_0 = 0, \qquad \Omega_1 = 0$$

$$3. \qquad 8w_2 u_0 w_0 + 9u_1^2 + 4w_1(\Omega_1 + 3u_1 w_0) = 0$$

$$4Au_2 Y_0 - 2Aw_2(2B_1 u_0 - BCw_0) + 4Au_1(u_1 - B_1 w_1) + \\ + ABCw_1^2 - v_0^2(4\Sigma_0 - ABC) = 0$$

$$2A\Omega_2 + (-4u_0 + 3Aw_0)v_0^2 = 0$$

$$4. \qquad -2u_3 u_0 + 2w_3 u_0 w_0 + u_2(3u_1 + 2w_1 w_0) + 2w_2 \Xi_1 + u_1 w_1^2 = 0$$

$$Au_3 Y_0 - Aw_3 \Gamma_0 + Au_2 Y_1 - Aw_2 \Gamma_1 - Ju_1 v_0^2 - 2v_1 \Sigma_0 v_0 - Nw_1 v_0^2 = 0$$

$$3A\Omega_3 - Au_2 w_1 + 3Aw_2 u_1 - u_1(3v_0^2 + Aw_0) + v_1(-8u_0 + 7Aw_0)v_0 + 2Aw_1 v_0^2 = 0$$

$$5. \qquad 2u_4 w_0^2 + 4(w_4 - h)u_0 w_0 + u_3(3u_1 + 4w_1 w_0) + 4w_3 \Xi_1 + 2u_2 \Sigma + \\ + 2w_2(w_2 u_0 + 2u_1 w_1) = 0$$

$$2A^2 u_4 Y_0 - 2A^2 w_4 \Gamma_0 + 2A^2 u_3 Y_1 - 2A^2 w_3 \Gamma_1 + 2A^2 u_2 Y_2 - \\ - 2AJu_2 v_0^2 - Av_2(4v_0\Sigma_0 + ABCv_0) + 2Aw_2(ABCw_2 - Nv_0^2) - \\ - 4AJu_1 v_1 v_0 - 2Av_1^2 \Sigma_0 - 4ANv_1 w_1 v_0 - 2P'v_0^4 = 0$$

$$4A\Omega_4 - 2Au_3 w_1 + 2Aw_3 u_1 - 2u_2 v_0^2 - 8v_2(u_0 - Aw_0)v_0 + \\ + Aw_2 v_0^2 - 6u_1 v_1 v_0 - 4v_1^2(u_0 - Aw_0) + 5Av_1 w_1 v_0 = 0$$

$$6. \qquad -6u_5 u_0 + 2w_5 u_0 w_0 + 2(u_4 w_1 + w_4 u_1)w_0 + 2(w_4 - h)w_1 u_0 + \\ + u_3 \Sigma + 2w_3 \Xi_2 + 2u_2 w_2 w_1 + w_2^2 u_1 = 0$$

$$A^2 u_5 Y_0 - A^2 w_5 \Gamma_0 + A^2 u_4 Y_1 - A^2 w_4 \Gamma_1 + Au_3(2Au_2 - AB_1 w_2 - Jv_0^2) - \\ - Av_3(2\Sigma_0 + ABC)v_0 - Aw_3(AB_1 u_2 - 2ABCw_2 + Nv_0^2) - 2AJu_2 v_1 v_0 - \\ - 2Av_2(\Sigma_1 v_0 + v_1 \Sigma_0) - 2ANw_2 v_1 v_0 - Av_1^2 \Sigma_1 - 4P'v_1 v_0^3 = 0$$

$$5A\Omega_5 - 3A(u_4 w_1 - w_4 u_1) - u_3(v_0^2 + Aw_2) - v_3(8u_0 - 9Aw_0)v_0 - \\ - Aw_3 u_2 - 4u_2 v_1 v_0 - v_2[6u_1 v_0 + v_1(8u_0 - 9Aw_0) - 6Aw_1 v_0] + \\ + 3Aw_2 v_1 v_0 - 3v_1^2(u_1 - Aw_1) = 0$$

. . . .

Here, for brevity, we put

$$\Gamma_0 = B_1 u_0 - 2BCw_0, \qquad \Gamma_1 = B_1 u_1 - 2BCw_1$$

$$\Xi_1 = u_1 w_0 + w_1 u_0, \qquad \Xi_2 = u_2 w_0 + w_2 u_0 + u_1 w_1$$

$$\Sigma = u_2 + 2w_2 w_0 + w_1^2, \qquad \Sigma_0 = Ju_0 + Nw_0, \qquad \Sigma_1 = Ju_1 + Nw_1$$

$$Y_0 = 2u_0 - B_1 w_0, \qquad Y_1 = 2u_1 - B_1 w_1, \qquad Y_2 = u_2 - B_1 w_2$$

$$\Omega_s = -u_s w_0 + w_s u_0 \qquad (s = 1, \ldots, 5)$$

The values

$$u_0 = -4B^2, \qquad w_0 = -4B \quad \text{or} \quad u_0 = -4C^2, \quad w_0 = -4C$$

are a solution of the first system of (3.4).

For precision we take the first values of u_0, w_0. The second system of (3.4) yields $u_1 = w_1 = 0$. The third system of (3.4) assumes the form

$$w_2 = 0$$
$$16A(B-C)u_2 - (16AB - 15AC - 16B^2 + 16BC)v_0^2 = 0$$
$$2Au_2 - (3A - 4B)v_0^2 = 0$$

If the condition $A = C = 4B/3$ previously obtained ([1], Section 5) is excluded, then for $v_0 \neq 0$, the determinant of the system consisting of the last two equations must be put equal to zero. This gives a necessary condition, which can be written in one of the following forms:

$$8AB - 9AC - 16B(B-C) = 0, \quad AC = 8(A-2B)(B-C),$$
$$C = \frac{8B(A-2B)}{9A-16B} \tag{3.5}$$

It follows that $A \neq 2B$, $B \neq C$, and

$$u_2 = \frac{3A-4B}{2A} v_0^2 \tag{3.6}$$

The fourth system of (3.4) becomes

$$\begin{gathered} u_3 + 4Bw_3 = 0 \\ Au_3 - ABw_3 - 2(A-B)v_1v_0 = 0 \\ 3A(u_3 - Bw_3) - (7A - 8B)v_1v_0 = 0 \end{gathered} \tag{3.7}$$

By virtue of (3.5), the solution of this system is

$$u_3 = w_3 = v_1 = 0$$

Because of the relations previously obtained the fifth system of (3.4) can be reduced to

$$16B^2(u_4 + 2Bw_4 - 2Bh) + \frac{(3A-4B)^2}{4A^2} v_0^4 = 0 \tag{3.8}$$
$$8A^2B(u_4 - Bw_4 - v_2v_0) - \frac{1}{2(B-C)}(A^2 - 6AB + 2AC + 8B^2 - 4BC)v_0^4 = 0$$
$$8B[A(u_4 - Bw_4) - 2(A-B)v_2v_0] - \frac{(3A-4B)}{2A} v_0^4 = 0$$

Using the third equality of (3.5), the last two equations of (3.8) yield

$$\frac{v_2}{v_0^3}=\frac{9A^3-57A^2B+108AB^2-64B^3}{16AB(A-2B)(9A-16B)(B-C)}$$

We will not derive explicit expressions for u_4, w_4.

The sixth system of (3.4) can similarly be reduced to

$$\begin{gathered}3u_5+4Bw_5=0\\ A(u_5-Bw_5)-2Bv_3v_0=0\\ 5A(u_5-Bw_5)-(9A-8B)v_3v_0=0\end{gathered}\tag{3.9}$$

The determinant of this system (relative to u_5, w_5, v_3) does not vanish because of (3.5); consequently, $u_5=w_5=v_3=0$, etc.

We assume, as before, that

$$\begin{aligned}\nu&=m_0+m_1\rho+m_2\rho^2+\ldots+m_n\rho^n\\ \tau&=n_0+n_1\rho+n_2\rho^2+\ldots+n_n\rho^n\end{aligned}\tag{3.10}$$

Substituting (3.3), with the values of u_i, v_i, w_i just found, in the left and right sides of (3.10), and equating coefficients of like powers of ξ, we obtain a system of algebraic equations for determining the coefficients m_i and n_i. Substituting the values of u_0, w_0, u_2, v_2 into the algebraic equations, we get

$$\begin{gathered}Bn_4-m_4=0,\quad Cn_4-m_4=\frac{4B(B-C)}{v_0^4}\\ Bn_2-m_2=-\frac{u_2}{v_0^2}=\frac{4B-3A}{2A}\\ Cn_2-m_2=-\frac{16B(B-C)v_2}{v_0^3}-\frac{u_2}{v_0^2}=-\frac{(3A-4B)(3A-8B)(5A-8B)}{2A(A-2B)(9A-16B)}\\ m_1=m_3=n_1=n_3=0,\qquad m_s=n_s=0\quad(s=5,6,7,\ldots)\end{gathered}\tag{3.11}$$

Substituting the expressions for ν and τ given by (3.10) into (1.6), with m_i and n_i as in (3.11), we determine

$$\begin{aligned}Q\gamma_2&=\frac{4B-3A}{2}pq\\ Q\gamma_3&=\left[-\frac{(3A-4B)(3A-8B)(5A-8B)}{2(A-2B)(9A-16B)}+\frac{8AB(B-C)}{v_0^4}p^2\right]rp\end{aligned}$$

It is not necessary to proceed further with the calculations to see that we have arrived at Goriachev's second case [10].

4. Other methods of obtaining the initial conditions of Steklov's and Goriachev's second cases. The initial conditions of Steklov's and Goriachev's second cases can and subsequently will be obtained by methods other than those they used. We will also investigate the possibility of supplementing these initial conditions in various ways, as well as that of integrating the resulting equations.

In his first paper [3], Field, denoting the principal moments of inertia by J_1, J_2, J_3, finds a special solution under the conditions

$$f = g = 0, \qquad k = 0$$
$$J_3^2 = 2J_1J_2, \qquad T = -\frac{2MghJ_3(J_1 + J_2 - J_3)}{(J_1 - J_3)(J_2 - J_3)} \tag{4.1}$$

with the additional restriction $J_1 > 2J_2$ or $J_2 > 2J_2$, where h, f, g are the coordinates of the center of gravity and T is a constant of the kinetic energy integral.

Gorliss, in his first paper, using the same notation as Field, finds the initial conditions for the special solution in the form

$$f = g = 0, \qquad k = 0$$
$$E = \frac{Mgh(2J_1J_2 - 2J_1J_3 - 2J_2J_3 + J_3^2)}{(J_3 - J_1)(J_3 - J_2)}, \quad 2J_2 < J_3 < J_1 \tag{4.2}$$

where E is a constant of the kinetic energy integral.

If we put $J_3^2 \quad 2J_1J_2$ in (4.2), we obtain conditions (4.1), assuming that the first supplementary condition $J_1 > 2J_2$ holds good.

To compare the conditions, we introduce the Table S (see below). In this Table symbols occurring in the same column can be substituted for each other. The first row is our notation, the second rowSteklov's, the third Field's, the fourth Gorliss's, the fifth Fabbri's, and the sixth Kuz'min's.

$$S\left\{\begin{array}{llllllllllllll}
(A, & B, & C, & x_0, & y_0, & z_0, & h, & k, & p, & q, & r, & \gamma_1, & \gamma_2, & \gamma_3) \\
(A, & B, & C, & \alpha, & 0, & 0, & -- & -- & p, & q, & r, & \xi, & \eta, & \zeta) \\
(J_3, & J_1, & J_2, & h, & 0, & 0, & T, & k, & \omega_3, & \omega_1, & \omega_2, & -\gamma_3, & -\gamma_1, & -\gamma_2) \\
(J_3, & J_1, & J_2, & h, & f, & g, & E, & k, & \omega_3, & \omega_1, & \omega_2, & -\gamma_3, & -\gamma_1, & -\gamma_2) \\
(C, & A, & B, & z_0, & 0, & 0, & m_1, & K_\xi & r, & p, & q, & \gamma_3, & \gamma_1, & \gamma_2) \\
(A, & B, & C, & x_0, & 0, & 0, & -- & -- & p, & q, & r, & -\gamma_1, & -\gamma_2, & -\gamma_3)
\end{array}\right.$$

In the Table the last three columns denote the direction cosines of the force of gravity relative to a fixed coordinate system:

$$\gamma_1 = \sin\theta \sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta, \quad m_1 = 2G\omega_0^2$$

Applying the substitution Table S, conditions (4.2) can be written in the form

$$2C < A < B, \quad y_0 = z_0 = 0, \quad x_0 \neq 0, \quad k = 0$$

$$h = \frac{Q(A^2 - 2AB - 2AC + 2BC)}{(A-B)(A-C)} \tag{4.3}$$

The constant of the area integral in Steklov's second case, $k = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3$ vanishes according to (2.5), since it is proportional to the integral which Steklov obtainedin the form

$$Ap^2 + (2B - A)q^2 + (2C - A)r^2 = 0 \tag{4.4}$$

Indeed, according to Steklov's formula

$$\begin{aligned} Q\gamma_2 &= \frac{(B-A)(C-A)}{2C-A} pq \\ Q\gamma_3 &= \frac{(B-A)(C-A)}{2B-A} pr \\ Q\gamma_1 &= \frac{(B-A)(C-A)}{(2B-A)(2C-A)} \left[Cp^2 - \frac{(2B-A)(B-C)}{A} q^2\right] \end{aligned} \tag{4.5}$$

or

$$Q\gamma_1 = \frac{(B-A)(C-A)}{(2B-A)(2C-A)} \left[Bp^2 + \frac{(2C-A)(B-C)}{A} r^2\right]$$

it follows that

$$k = \frac{(A-B)(A-C)Cp}{Q(2B-A)(2C-A)} [Ap^2 + (2B-A)q^2 + (2C-A)r^2]$$

which, because of (4.4), gives $k = 0$.

By (4.4), Steklov's expression for γ_1 can be rewritten as

$$Q\gamma_1 = -\frac{(B-A)(C-A)}{A(2B-A)(2C-A)} [B(2B-A)q^2 + C(2C-A)r^2] \tag{4.6}$$

and this coincides with the value for γ_1 given by (2.5). Steklov obtained yet another integral

$$A^2p^2 + B(2B-A)q^2 + C(2C-A)r^2 = K_1 \tag{4.7}$$

where the constant K_1 is determined from the relation

$$\frac{K_1^2(B-A)^2(C-A)^2}{Q^2A^2(2B-A)^2(2C-A)^2} = 1$$

The constant of the energy integral $h = Ap^2 + Bq^2 + Cr^2$ according to (4.4), (4.6), and (4.7) is

$$h = \varepsilon_0 \frac{Q(A^2 - 2AB - 2AC + 2BC)}{(B - A)(C - A)} \qquad (\varepsilon_0 = \pm 1) \tag{4.8}$$

Comparing condition (4.8) and the restriction imposed on A, B, C, y_0, z_0, k with condition (4.3), we see that they coincide (for $\epsilon_0 = +1$).

In his second paper [4], Gorliss, generalizing his previous case (including Field's) and Kowalewski's, found the initial conditions for the other special solution in the form

$$\begin{gathered} f = g = 0, \qquad k = 0, \qquad J_2 = \frac{J_1(16J_1 - 8J_3)}{16J_1 - 9J_3} \\ E = \frac{4Mgh(J_3 - 2J_1)(64J_1^2 - 56J_1J_3 - 9J_3^2)}{(4J_1 - 3J_3)(64J_1^2 - 64J_1J_3 + 15J_3^2)} \end{gathered} \tag{4.9}$$

Applying Table S, condition (4.9) can be written as

$$\begin{gathered} C = \frac{8B(A - 2B)}{9A - 16B}, \qquad y_0 = z_0 = 0, \qquad k = 0 \\ h = \frac{4Q(A - 2B)(9A^2 - 56AB + 64B^2)}{(3A - 4B)(15A^2 - 64AB + 64B^2)} \end{gathered} \tag{4.10}$$

The constant of the area integral $k = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3$ in Goriachev's second case is equal to zero (as the coefficient of the O-th power of p) according to the formulas he obtained for γ_1, γ_2, γ_3.

By virtue of Goriachev's relations [10]

$$\begin{gathered} \frac{B - C}{A} r^2 - \frac{A - B + \chi}{C} p^2 = L \\ Q\gamma_1 = 4l_2 \frac{A - B + \chi}{C} p^4 + \left(\chi + 3l_2L + 2\lambda \frac{A - B + \chi}{C}\right) p^2 + L\lambda \\ l_2\left(1 + \frac{\chi}{B} - 4\frac{A - B + \chi}{C}\right) p^4 + \left(\lambda \frac{C - 2A + 2B - 2\chi}{C} - \chi \frac{B + C - A - \lambda}{B} - \right. \\ \left. - 3l_2L\right) p^2 - \chi \frac{B - C}{A} q^2 - L\lambda = 0 \end{gathered} \tag{4.11}$$

where

$$\begin{gathered} L = \frac{QBC}{(3A - 4B)(2B - C)(2B - 3C)}, \qquad \chi = \frac{4B - 3A}{2} \\ \lambda = \frac{(3A - 4B)(2B - C)(2B - 3C)}{BC} \end{gathered} \tag{4.12}$$

the constant of the energy integral is a polynomial of fourth degree in p. Putting the constant terms in the polynomial in p equal to zero, we get

$$h=\left[-\frac{AB\lambda}{(B-C)\chi}+\frac{AC}{B-C}-2\lambda\right]L$$

By the third relation of (3.5) and (4.12) the constant h has the value indicated in (4.10).

The restrictions on A, B, C, y_0, z_0, h obtained by Gorliss are the same as those in Goriachev's second case.

5. Kowalewski's case. In the system (1.1) we perform the transformation $t = \eta^{3/2}$ on the independent variable. Then (1.1) reduces to

$$\left(\frac{d\nu}{d\eta}\right)^2=\frac{9\eta}{4}\{-\nu(\tau-h)^2+4Q[Q(\nu-\rho^2)+k\rho(\tau-h)-Qk^2]\}$$
$$A^2BC\left(\frac{d\rho}{d\eta}\right)^2=\frac{9\eta}{4}(-A^2\nu^2+AJ\nu\rho^2+A^2B_1\nu\tau+P'\rho^4+AN\rho^2\tau-A^2BC\tau^2) \qquad (5.1)$$
$$(\rho^2-A\tau)\frac{d\nu}{d\eta}+A[\rho(\tau-h)-2Qk]\frac{d\rho}{d\eta}+A(\nu-\rho^2)\frac{d\tau}{d\eta}=0$$

The series

$$\begin{aligned}\nu&=\eta^{-3}(e_0+e_1\eta+e_2\eta^2+\ldots) && (e_0\neq 0)\\ \rho&=\eta^{-1}(s_0+s_1\eta+s_2\eta^2+\ldots) && (s_0\neq 0)\\ \tau&=\eta^{-3}(\varkappa_0+\varkappa_1\eta+\varkappa_2\eta^2+\ldots) && (\varkappa_0\neq 0)\end{aligned} \qquad (5.2)$$

satisfy (5.1) formally.

Substituting the series (5.2) in equations (5.1), equating coefficients of equal power of η and assuming that $k = 0$ (one of the necessary conditions for the existence of a unique solution obtained in [1]), we get the following system of equations for determining the coefficients of the series (5.2):

1. $\quad 4e_0+\varkappa_0^2=0, \qquad e_0^2-B_1e_0\varkappa_0+BC\varkappa_0^2=0$

(The third equation is an identity).

2. $\quad e_1(16e_0+3\varkappa_0^2)+6\varkappa_1e_0\varkappa_0=0$

$$9Ae_1(2e_0-B_1\varkappa_0)-9A\varkappa_1(B_1e_0-2BC\varkappa_0)+s_0^2[4ABC-9(Je_0+N\varkappa_0)]=0$$
$$A(e_1\varkappa_0-\varkappa_1e_0)+3e_0s_0^2-2As_0^2\varkappa_0=0$$

3. $\quad 3e_2(8e_0+3\varkappa_0^2)+18\varkappa_2e_0\varkappa_0+16e_1^2+18e_1\varkappa_1\varkappa_0+9\varkappa_1^2e_0=0$

$$A^2e_2(2e_0-B_1\varkappa_0)-A^2\varkappa_2(B_1e_0-2BC\varkappa_0)+A^2e_1^2-Ae_1(AB_1\varkappa_1+Js_0^2)-$$
$$-2As_1(Je_0+N\varkappa_0)s_0+A^2BC\varkappa_1^2-AN\varkappa_1s_0^2-P's_0^4=0$$
$$2A(e_2\varkappa_0+\varkappa_2e_0)-2e_1s_0^2-s_1(6e_0-5A\varkappa_0)s_0+A\varkappa_1s_0^2=0 \qquad (5.3)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

The solution of the first system of (5.3) is

$$e_0 = -4B^2, \quad \varkappa_0 = -4B \quad \text{or} \quad e_0 = -4C^2, \quad \varkappa_0 = -4C$$

For precision, we will take the first values.

The second system of (5.3) assumes the form

$$\begin{gathered} e_1 - 6B\varkappa_1 = 0 \\ 9A(B-C)(e_1 - B\varkappa_1) - (9AB - 8AC - 9B^2 + 9BC)s_0^2 = 0 \\ A(e_1 - B\varkappa_1) - (2A - 3B)s_0^2 = 0 \end{gathered}$$

To have $s_0 \neq 0$ the determinant of the system relative to e_1, κ_1, s_0 must be put equal to zero, This gives

$$A = \frac{18B(B-C)}{9B-10C} \tag{5.4}$$

Then

$$e_1 = \frac{6(2A-3B)s_0^2}{5A}, \qquad \varkappa_1 = \frac{(2A-3B)s_0^2}{5AB}$$

Further computation of the coefficients is not necessary. We assume, as we did earlier, that

$$\nu = \chi_0 + \chi_1\rho + \chi_2\rho^2 + \ldots + \chi_n\rho^n, \quad \tau = \psi_0 + \psi_1\rho + \psi_2\rho^2 + \ldots + \psi_n\rho^n \tag{5.5}$$

Substituting the series (5.2) into (5.5) and equating coefficients of like powers of η, we get a system of algebraic equations for determining the χ_i and ϕ_i.

Putting $e_0 = -4B^2$, $\kappa_0 = -4B$ in these equations, we get

$$\begin{gathered} B\psi_3 - \chi_3 = 0, \qquad C\psi_3 - \chi_3 = \frac{4B(B-C)}{s_0^2} \\ B\psi_2 - \chi_2 = \frac{3B-2A}{A}, \qquad C\psi_2 - \chi_2 = \frac{(2A-3B)(C-6B)}{5AB} - \frac{12B(B-C)s_1}{s_0^3} \\ B\psi_1 - \chi_1 = \frac{B\varkappa_2 - e_2}{s_0} + \frac{2(2A-3B)s_1}{A} \\ C\psi_1 - \chi_1 = \frac{C\varkappa_2 - e_2}{s_0} - 2(C\psi_2 - \chi_2)s_1 - 3(C\psi_3 - \chi_3)(s_2 s_0 + s_1^2) \\ \psi_i = \chi_i = 0 \qquad (i = 4, 5, 6, \ldots) \end{gathered} \tag{5.6}$$

Substituting the expressions (5.5) for ν, τ into (1.6), we obtain

$$\begin{aligned} 2Q\gamma_2 &= q\,[B\psi_1 - \chi_1 + 2(B\psi_2 - \chi_2)\rho + 3(B\psi_3 - \chi_3)\rho^2] \\ 2Q\gamma_3 &= r\,[C\psi_1 - \chi_1 + 2(C\psi_2 - \chi_2)\rho + 3(C\psi_3 - \chi_3)\rho^2] \end{aligned} \tag{5.7}$$

Kowalewski [11] found the condition on A, B, C expressed by (5.4) and expressions for γ_2, γ_3 given by

$$\begin{aligned} 2Q\gamma_2 &= -C\left[\beta_1 + 2\left(\beta_2 - \frac{A-B}{C}\right)p\right]q \\ 2Q\gamma_3 &= B\left[\alpha_1'\beta_1 + 2\left(\alpha_2 - \frac{C-A}{B}\right)p + 3\alpha_3'\beta_1^{-1}p^2\right]r \end{aligned} \tag{5.8}$$

For example, we compare the coefficient of pq in the expression for γ_2 in Kowalewski's formula (5.8) with that of (5.7).

From (5.8), this coefficient, by (5.4), is equal to

$$\frac{B(9B-6C)}{Q(9B-10C)} \tag{5.9}$$

The same coefficient, according to (5.7), (5.4) and (5.6) is equal to the same expression, with sign reversed. The difference in sign is explained by the fact that the sign of the factor Q in Euler's equations differs in the cases being compared. Hence (5.4) and (5.7) characterize Kowalewski's case.

6. Chaplygin's second case. We consider the transformations

$$\rho = z^{3/2}, \qquad t = \zeta^{3/2}$$

for ρ and t in (1.1) and assume that the condition $k = 0$ is satisfied.

After these transformations, (1.1) transforms into

$$\begin{aligned} \left(\frac{d\nu}{d\zeta}\right)^2 &= \frac{9\zeta}{4}\left[-\nu(\tau-h)^2 + 4Q^2(\nu - z^3)\right] \\ A^2BCz\left(\frac{dz}{d\zeta}\right)^2 &= \zeta\left[-A^2\nu^2 + AJ\nu z^3 + A^2B_1\nu\tau + P'z^6 + ANz^3\tau - A^2BC\tau^2\right] \\ (z^3 - A\tau)\frac{d\nu}{d\zeta} &+ \frac{3}{2}A(\tau-h)z^2\frac{dz}{d\zeta} + A(\nu - z^3)\frac{d\tau}{d\zeta} = 0 \end{aligned} \tag{6.1}$$

This system is formally satisfied by the series

$$\begin{aligned} \nu &= \zeta^{-3}(g_0 + g_1\zeta + g_2\zeta^2 + \ldots) \qquad (g_0 \neq 0) \\ z &= \zeta^{-1}(h_0 + h_1\zeta + h_2\zeta^2 + \ldots) \qquad (h_0 \neq 0) \\ \tau &= \zeta^{-3}(l_0 + l_1\zeta + l_2\zeta^2 + \ldots) \qquad (l_0 \neq 0) \end{aligned} \tag{6.2}$$

Substituting the series (6.2) into (6.1) and comparing coefficients of equal powers of ζ, we find the following systems for determining the coefficients of the series (6.2):

1.
$$\begin{gathered} 4g_0 + l_0'^2 = 0 \\ A^2g_0^2 - Ag_0(Jh_0^3 + AB_1l_0) - P'h_0^6 - Ah_0^3(Nl_0 - ABC) + A^2BCl_0^2 = 0 \\ 2g_0 - Al_0 = 0 \end{gathered}$$

$$2. \qquad g_1 l_0^2 - 6 l_1 g_0 l_0 = 0$$

$$Ag_1(2Ag_0 - Jh_0^3 - AB_1 l_0) - h_1(3AJg_0 + 6P'h_0^3 + 3ANl_0 - A^2BC)h_0^2 - \\ - Al_1(AB_1 g_0 + Nh_0^3 - 2ABCl_0) = 0$$

$$2g_1(2h_0^3 + Al_0) + 6h_1(3g_0 - 2Al_0)h_0^2 - Al_1(2g_0 + h_0^3) = 0$$

$$3. \qquad 3g_2 l_0^2 + 18 l_2 g_0 l_0 + 16 g_1^2 + 18 g_1 l_1 l_0 + 9 l_1^2 g_0 = 0$$

$$Ag_2(2Ag_0 - Jh_0^3 - AB_1 l_0) - h_2(3AJg_0 + 6P'h_0^3 + 3ANl_0 + A^2BC)h_0^2 - \\ - Al_2(AB_1 g_0 + Nh_0^3 - 2ABCl_0) + A^2 g_1^2 - Ag_1(3Jh_1h_0^2 + AB_1 l_1) - \\ - h_1^2(3AJg_0 + 15P'h_0^3 + 3ANl_0)h_0 + A^2BCl_1^2 - 3ANh_1 l_1 h_0^2 = 0$$

$$2g_2(h_0^3 + 2Al_0) + 3h_2(6g_0 - 5Al_0)h_0^2 - Al_2(4g_0 - h_0^3) + \\ + 12 g_1 h_1 h_0^2 + 3h_1^2(6g_0 - 5Al_0)h_0 - 6Ah_1 l_1 h_0^2 = 0 \qquad (6.3)$$

. . . .

A solution of the first system of (6.3) is

$$g_0 = -A^2, \qquad h_0^3 = -\frac{A^2(A-2B)(A-2C)}{(A-B)(A-C)}, \qquad l_0 = -2A \qquad (6.4)$$

The second system of (6.3) takes the form

$$g_1 - 3Al_1 = 0$$

$$Ag_1[Jh_0^3 + 2A^2(A - B - C)] + h_1[6P'h_0^3 - 6A^4 + 9A^3B_1 - 13A^2BC]h_0^2 + \\ + l_1[Nh_0^3 - A^3B_1 + 4A^2BC] = 0$$

$$4g_1(h_0^3 - A^2) + 6A^2h_1h_0^2 - Al_1(h_0^3 - 2A^2) = 0$$

If the determinant of the system relative to g_1, h_1, l_1 does not vanish, then

$$g_1 = h_1 = l_1 = 0 \qquad (6.5)$$

The third system of (6.3) assumes the form

$$g_2 + 3Al_2 = 0$$

$$Ag_2[Jh_0^3 + 2A^2(A - B_1)] + h_2(6P'h_0^3 - 6A^4 + 9A^3B_1 - 11A^2BC)h_0^2 + \\ + Al_2(Nh_0^3 - A^3B_1 + 4A^2BC) = 0$$

$$2g_2(h_0^3 - 4A^2) + 12A^2h_2h_0^2 + Al_2(h_0^3 + 4A^2) = 0$$

Because of the value of h_0^3 from (6.4) the determinant of the system relative to g_2, h_2, l_2 is equal to

$$\Delta = 7A^4 \sqrt[3]{\frac{A^4(A-2B)^2(A-2C)^2}{(A-B)^5(C-A)^5}}\,(AB_1 - 3BC)(9A^2 - 18AB_1 + 32BC)$$

If we assume that $l_2 \neq 0$, then necessarily $\Delta = 0$, and this yields the condition

$$4BC = 9(A - 2B)(A - 2C) \qquad (6.6)$$

Without computing the coefficients further, we suppose, as before, that

$$\nu = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \ldots + \lambda_n z^n, \qquad \tau = \pi_0 + \pi_1 z + \pi_2 z^2 + \ldots + \pi_n z^n \tag{6.7}$$

Substituting the series (6.2) into (6.7) and equating coefficients of equal powers of ζ we obtain a system of algebraic equations for determining λ_i, π_i.

Using (6.5) and (6.4), we get from these algebraic equations:

$$\begin{gathered}
B\pi_3 - \lambda_3 = \frac{(A-B)(C-A)}{A(A-2C)}, \qquad C\pi_3 - \lambda_3 = \frac{(A-B)(C-A)}{A(A-2B)} \\
B\pi_1 - \lambda_1 = \frac{Bl_2 - g_2}{h_0} - 3\frac{(A-B)(C-A)}{A(A-2C)} h_2 h_0 \\
C\pi_1 - \lambda_1 = \frac{Cl_2 - g_2}{h_0} - 3\frac{(A-B)(C-A)}{A(A-2B)} h_2 h_0 \\
\lambda_2 = \pi_2 = 0, \ \lambda_i = \pi_i = 0 \qquad (i = 4, 5, 6, \ldots)
\end{gathered} \tag{6.8}$$

Substituting the expressions for ν and r from (6.7) into (1.6) yields

$$\begin{aligned}
2Q\gamma_2 &= q\,[B\pi_1 - \lambda_1 + 2(B\pi_2 - \lambda_2)z + 3(B\pi_3 - \lambda_3)z^2]\,\frac{dz}{d\rho} \\
2Q\gamma_3 &= r\,[C\pi_1 - \lambda_1 + 2(C\pi_2 - \lambda_2)z + 3(C\pi_3 - \lambda_3)z^2]\,\frac{dz}{d\rho}
\end{aligned} \tag{6.9}$$

As an example, the coefficient of pq in the expression for γ_2 in (6.9) is

$$\frac{(A-B)(C-A)}{Q(A-2C)}$$

according to (6.8), if we go back to the old variables. This value is the same as that obtained by Chaplygin [12] in his second case.

It is not necessary to calculate or compare coefficients further, to assert that (6.9) characterizes Chaplygin's second case [12].

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